Calculus II - Day 17

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Goals for today:

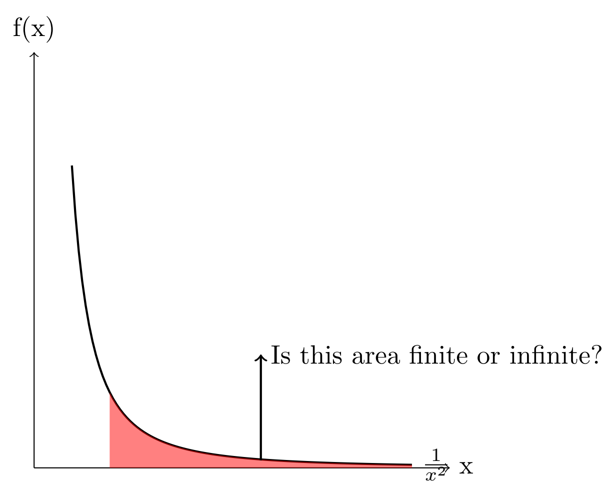
- Compute integrals of the form $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$
- Compute integrals of functions with vertical asymptotes
- Use the Comparison Test for integrals to determine whether an improper integral converges or diverges

Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Question: What if f is not continuous, or $a = -\infty$, or $b = \infty$?
Example: $\int_1^\infty \frac{1}{x^2}\,dx$



Does this integral "converge"?
We approach this by computing $\int_1^t \frac{1}{x^2}\,dx$, then taking:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = \boxed{1}$$

If $\int_a^t f(x)\,dx$ exists for all $t > a$, then:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{if this limit exists.}$$

If $\int_t^b f(x)\,dx$ exists for all $t < b$, then:

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx \quad \text{if this limit exists.}$$

If the limit above exists and is finite, we say the integral converges. If the limit is $\pm\infty$ or does not exist, we say the integral diverges.

If $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ both converge, we define:

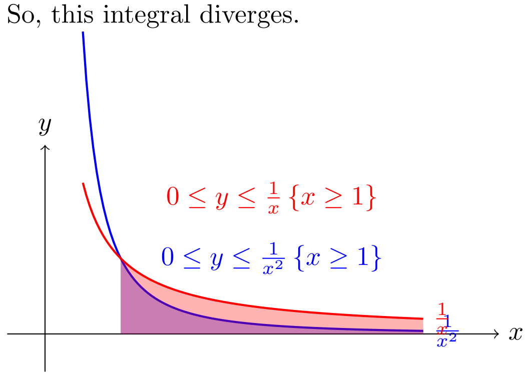
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

$\int_1^\infty \frac{1}{x^2}\,dx = 1$, so this integral converges.

Example:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}[\ln|x|]_1^t = \lim_{t\to\infty}(\ln|t| - \ln|1|) = \infty - 0 = \infty$$

So, this integral diverges.



Recall: p-test for series:

$$\sum_{k=1}^\infty \frac{1}{k^p}$$

- Converges if $p > 1$ (finite)
- Diverges if $p \le 1$ (∞)

p-test for integrals:
If $p > 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{(converges)}$$

If $0 \le p \le 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \infty \quad \text{(diverges)}$$

Example:

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx \quad u = x,\, du = dx$$

$$= \lim_{t\to-\infty}[xe^x]_t^0 - \int_t^0 e^x\,dx$$

$$= \lim_{t\to-\infty}[(xe^x - e^x)]_t^0$$

$$= \lim_{t\to-\infty}\left[(0-1) - (te^t - e^t)\right]$$

$$= \lim_{t\to-\infty}\left(-1 + e^t - te^t\right) = -1 + 0 - ??$$

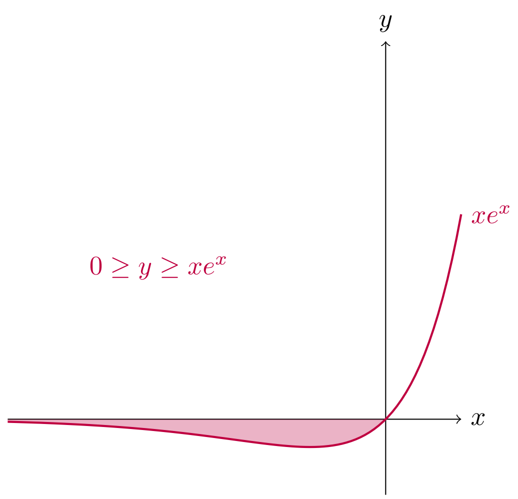
Note

Note: What is $\lim_{t\to-\infty} te^t$?

$$\lim_{t\to-\infty} te^t = \text{"} -\infty \cdot 0\text{"} = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \frac{-\infty}{\infty}$$

$$= \text{(L'Hopital's Rule)} = \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \frac{1}{-\infty} = 0$$

$$\lim_{t\to-\infty}\left(-1 + e^t - te^t\right) = -1 + 0 - 0 = \boxed{-1}$$



Example:

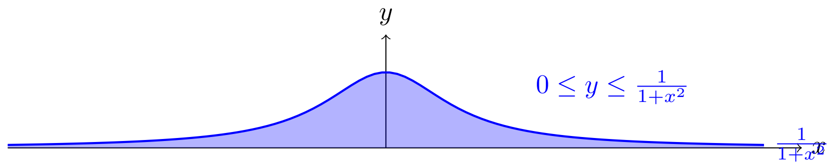
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{s\to-\infty} \int_{s}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{s\to-\infty} \arctan(x)\Big|_{s}^{0} + \lim_{t\to\infty} \arctan(x)\Big|_{0}^{t}$$

$$= \lim_{s\to-\infty} (0 - \arctan(s)) + \lim_{t\to\infty} (\arctan(t) - 0)$$

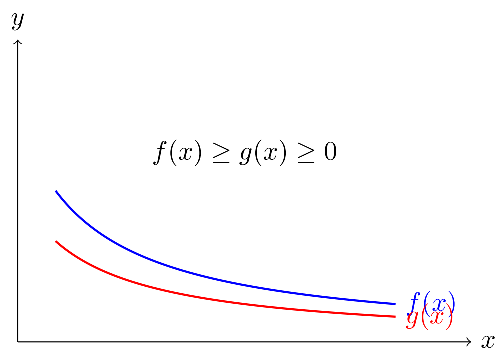
$$= -(-\pi/2) + \pi/2 = \boxed{\pi}$$



A comparison test for integrals:

Suppose f and g are two non-negative continuous functions on $[a,\infty)$ such that for all $x \ge a$:

$$f(x) \ge g(x) \ge 0$$



Then:

$$\int_a^\infty f(x)\,dx \ge \int_a^\infty g(x)\,dx$$

- If $\int_a^\infty f(x)\,dx$ converges, so does $\int_a^\infty g(x)\,dx$.
- If $\int_a^\infty g(x)\,dx$ diverges, so does $\int_a^\infty f(x)\,dx$.

Example: Show $\int_0^\infty e^{-x^2}\,dx$ converges

Comparison Test

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

On $[1, \infty)$, $e^{-x^2} \leq e^{-x}$.

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left(-e^{-x} \Big|_1^t \right) = \lim_{t\to\infty} \left(-e^{-t} + e^{-1} \right) = \frac{1}{e}$$

By the Comparison Test, since $\int_1^\infty e^{-x}\,dx \geq \int_1^\infty e^{-x^2}\,dx$ and the former converges, the latter does as well, so the area is finite.

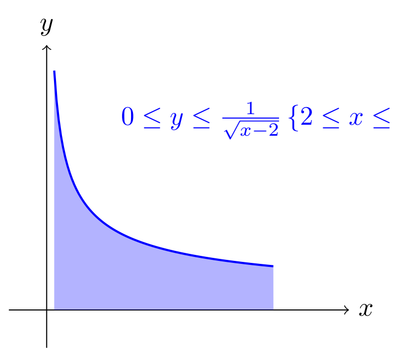
Vertical asymptotes:

Example: $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ There's a vertical asymptote at $x = 2$, so $f(x)$ is not continuous, and we can't use the Fundamental Theorem of Calculus.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx \quad u = x-2,\ du = dx,\ u(t) = t-2,\ u(5) = 5-2 = 3$$

$$= \lim_{t\to 2^+} \int_{t-2}^3 \frac{1}{\sqrt{u}}\,du = \lim_{t\to 2^+} \left[2\sqrt{u}\right]_{t-2}^3$$

$$= \lim_{t\to 2^+} \left(2\sqrt{3} - 2\sqrt{t-2} \right) = 2\sqrt{3}$$

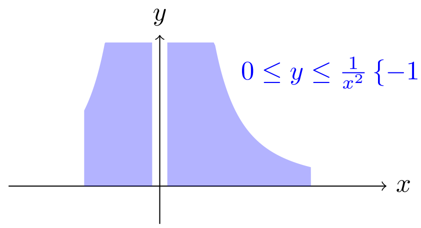


Example:

$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \left(-\frac{1}{x}\Big|_{-1}^{t} \right)$$

$$= \lim_{t\to 0^-} \left(-\frac{1}{t} + 1 \right) = \infty + 1 = \infty$$

(integral diverges)



Improper integrals pt. 2:

If f is continuous on $[a, b)$ and has a vertical asymptote at $x = b$:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

If f is continuous on $(a, b]$ and has a vertical asymptote at $x = a$:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

If f has a vertical asymptote at $x = c$ where $a < c < b$, but is otherwise continuous on $[a, b]$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{s\to c^-} \int_a^s f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$

In any of those cases, the integral converges if the limit is finite, and diverges if not.